

Funkcija dvije nezavisne promjenjive

Neka je S neprazan podskup prostora \mathbb{R}^2 i $T \subseteq \mathbb{R}$. Ako svakoj tački $M(x, y) \in S$ možemo unaprijed po datom pravilu f pridružiti jednu i samo jednu realnu vrijednost $z \in T$, tada kažemo da je data realna f -ja dvije realne promjenjive f iz \mathbb{R}^2 u \mathbb{R} (sa skupa $S \subseteq \mathbb{R}^2$ u skup $T \subseteq \mathbb{R}$) i pišemo $z = f(x, y)$. Skup S na kojem je određena f -ja f naziva se domen ili definiciono područje f -je f (označavat ćemo ga sa $D(f)$), a skup $f(A)$ skup vrijednosti f -je f ili kodomen (označavat ćemo ga sa $R(f)$). Ako za f -ju, zadana analitički (formulom) nije data oblast njene definisanosti, onda se pod njom podrazumjeva skup svih tačaka $M \in \mathbb{R}^2$ u kojoj f -ja, odnosno njen analitički izraz imaju određenu realnu vrijednost.

Za svaku od sljedećih f-ja, izračunati $f(3,2)$, i odrediti i skicirati domen.

a) $f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$

b) $f(x,y) = x \ln(y^2-x)$

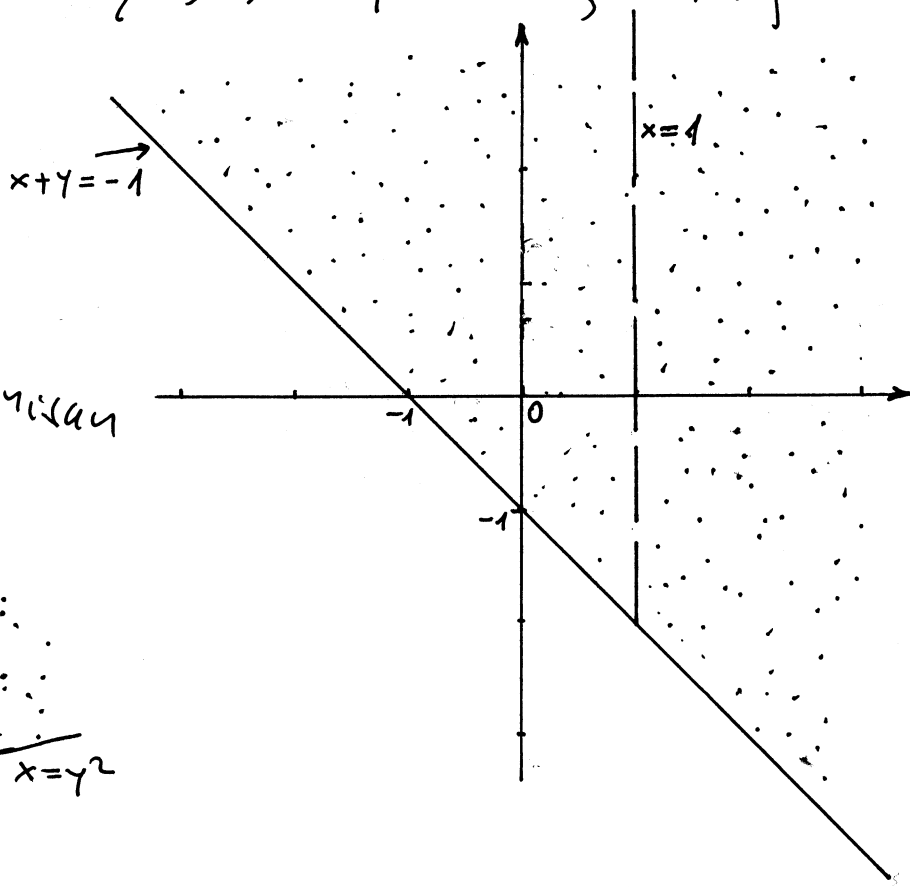
Rj. a) $f(3,2) = \frac{\sqrt{3+2+1}}{3-1} = \frac{\sqrt{6}}{2}$

Izraz za f-ju $f(x,y)$ ima smisla ako je nazivnik različit od nule i ako je vrijednost pod korijenom nenegativna:

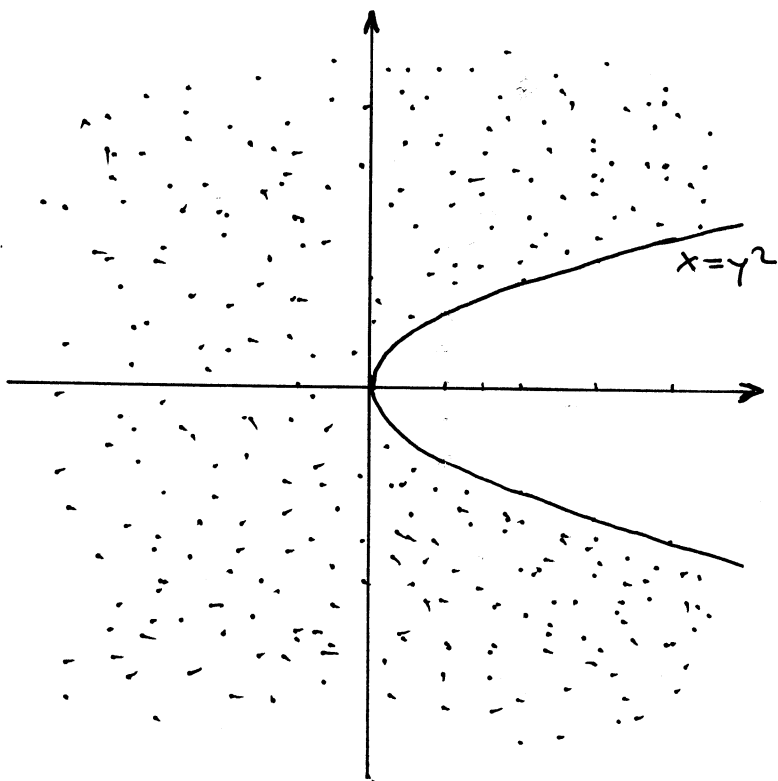
$$\begin{aligned} x-1 &\neq 0 & \Rightarrow & x \neq 1 \\ x+y+1 &\geq 0 & \Rightarrow & x+y \geq -1 \end{aligned}$$

Domen f-je f je $D = \{(x,y) \in \mathbb{R}^2 \mid x+y \geq -1, x \neq 1\}$

b) $f(3,2) = 3 \ln(2^2-3)$
 $= 3 \ln(4-3) = 3 \ln 1$
 $= 0$



Izraz $\ln(y^2-x)$ je definisan samo ako je $y^2-x > 0$



$$D = \{(x,y) \mid x < y^2\}$$

#) Odrediti domen i rang f-je $g(x,y) = \sqrt{9-x^2-y^2}$

Rj.
F-ja ima smisla akko $9-x^2-y^2 \geq 0$
 $x^2+y^2 \leq 9$

Domen f-je $g(x,y)$ je $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 9\}$

(znamo da je $x^2+y^2=9$ krug sa centrom u tački $C(0,0)$ poluprečnika $r=3$).

Rang f-je g je

$$\{z \in \mathbb{R} \mid z = \sqrt{9-x^2-y^2}, (x,y) \in D\}$$

Primjetimo da je

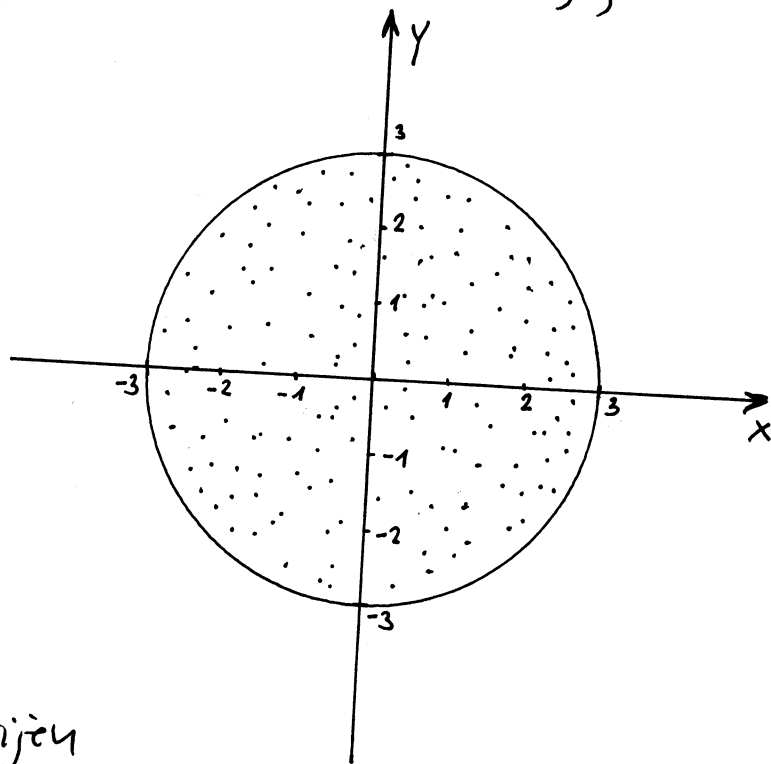
$$9-x^2-y^2 \leq 9 \text{ za } \forall (x,y) \in D$$

$$\text{pa je } \sqrt{9-x^2-y^2} \leq 3$$

z je pozitivan kvadratni korijen
 $z \geq 0$

Prema tome, rang f-je $g(x,y)$ je

$$\{z \mid 0 \leq z \leq 3\} = [0, 3]$$



#) Skicirati graf f-je $f(x,y) = 6 - 3x - 2y$.

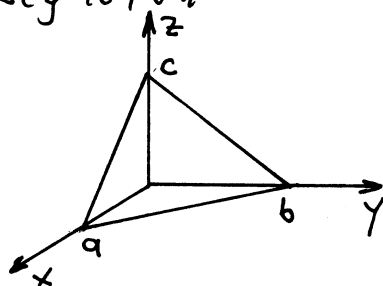
Rj. Graf f-je $f(x,y)$ ima jednačinu $z = 6 - 3x - 2y$

$$3x + 2y + z = 6$$

ovo predstavlja ravan.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

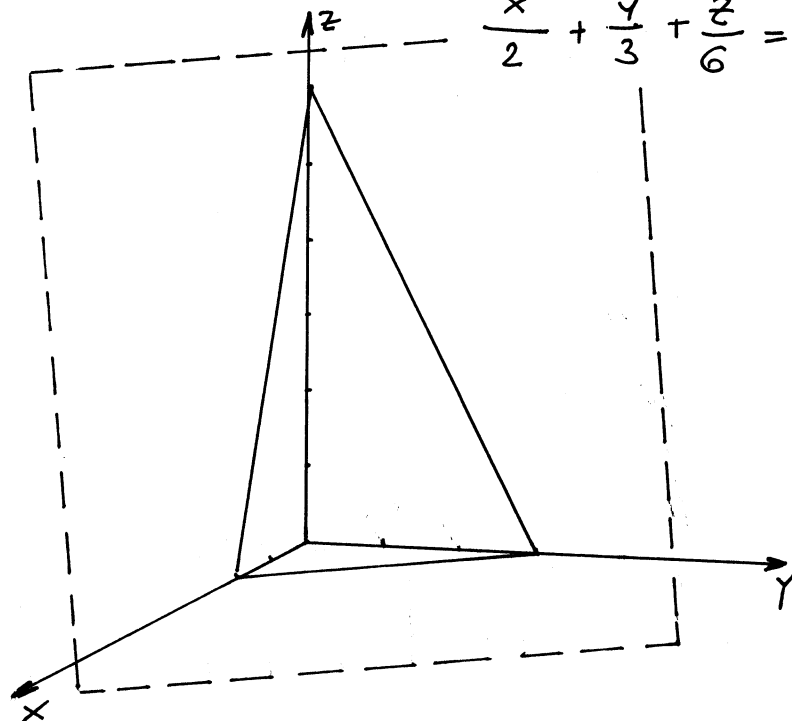
segmentni oblik jednačine ravni;



U našem slučaju

$$3x + 2y + z = 6 \quad | :6$$

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$



Ⓝ Skicirati graf f-je $g(x,y) = \sqrt{9-x^2-y^2}$.

kj. Graf f-je ima jednačinu $z = \sqrt{9-x^2-y^2}$

$$z = \sqrt{9-x^2-y^2} \quad |^2$$

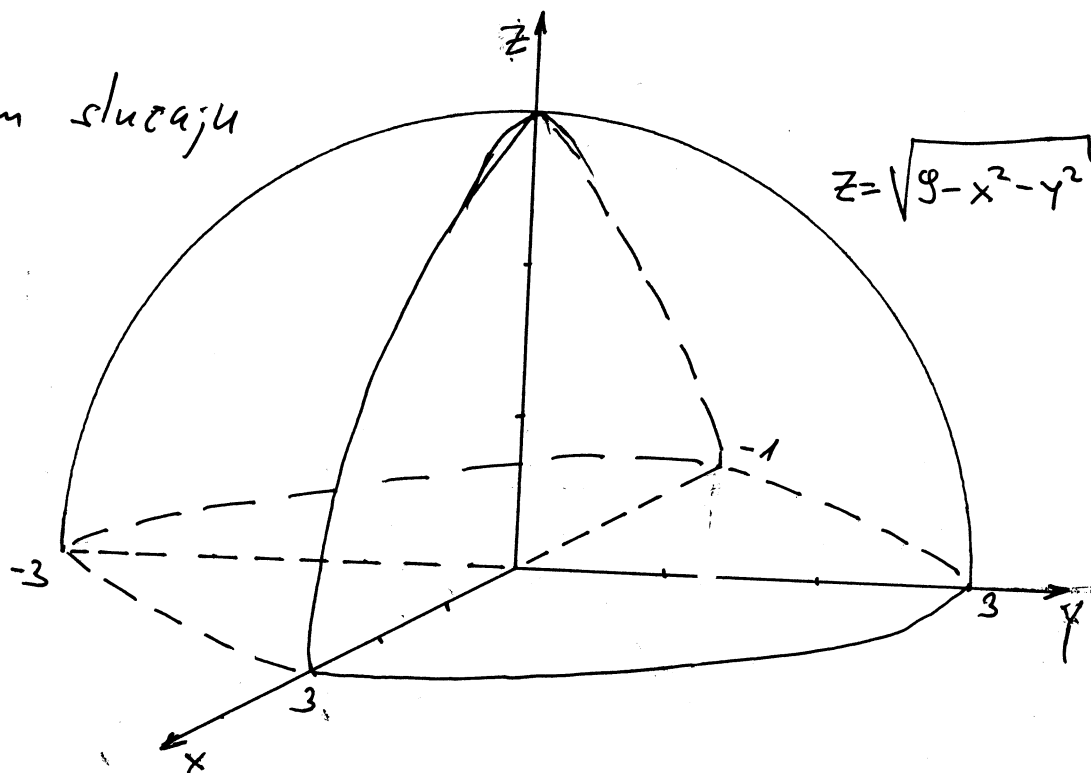
$$z^2 = 9-x^2-y^2$$

$$x^2+y^2+z^2 = 9$$

$$x^2+y^2+z^2 = R^2$$

je jednačina sfere sa
centrom u koordinatnom početku
poluprečnika R

U našem slučaju



Dio tablice izvoda

$$1) (c)' = 0;$$

$$3) (uv)' = u'v + v'u;$$

$$4) \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2};$$

$$4b) \left(\frac{c}{v}\right)' = -\frac{cv'}{v^2};$$

$$6) (\sin x)' = \cos x;$$

$$8) (\operatorname{tg} x)' = \sec^2 x;$$

$$2) (u + v - w)' = u' + v' - w';$$

$$3a) (cu)' = cu';$$

$$4a) \left(\frac{u}{c}\right)' = \frac{u'}{c};$$

$$5) (x^n)' = nx^{n-1};$$

$$7) (\cos x)' = -\sin x;$$

$$9) (\operatorname{ctg} x)' = -\operatorname{cosec}^2 x.$$

$$5) (u^n)' = nu^{n-1} \cdot u';$$

$$6) (\sin u)' = \cos u \cdot u';$$

$$7) (\cos u)' = -\sin u \cdot u';$$

$$8) (\operatorname{tg} u)' = \sec^2 u \cdot u';$$

$$9) (\operatorname{ctg} u)' = -\operatorname{cosec}^2 u \cdot u'.$$

$$10) (a^u)' = a^u \ln a \cdot u';$$

$$10a) (e^u)' = e^u u';$$

$$10b) (a^x)' = a^x \ln a;$$

$$10b) (e^x)' = e^x;$$

$$11) (\log u)' = \frac{u'}{u} \log e;$$

$$11a) (\ln u)' = \frac{u'}{u};$$

$$11b) (\log x)' = \frac{1}{x} \log e;$$

$$11b) (\ln x)' = \frac{1}{x}.$$

$$12) (\operatorname{arc} \sin u)' = \frac{u'}{\sqrt{1-u^2}};$$

$$13) (\operatorname{arc} \cos u)' = -\frac{u'}{\sqrt{1-u^2}};$$

$$14) (\operatorname{arc} \operatorname{tg} u)' = \frac{u'}{1+u^2};$$

$$15) (\operatorname{arc} \operatorname{ctg} u)' = -\frac{u'}{1+u^2};$$

$$12a) (\operatorname{arc} \sin x)' = \frac{1}{\sqrt{1-x^2}};$$

$$13a) (\operatorname{arc} \cos x)' = -\frac{1}{\sqrt{1-x^2}};$$

$$14a) (\operatorname{arc} \operatorname{tg} x)' = \frac{1}{1+x^2};$$

$$15a) (\operatorname{arc} \operatorname{ctg} x)' = -\frac{1}{1+x^2}.$$

Ekstremne vrijednosti f-ja dviju promjenjivih

Neka je data f-ja $z = f(x, y)$.

$$\begin{aligned}\frac{\partial z}{\partial x} &= 0 \\ \frac{\partial z}{\partial y} &= 0\end{aligned}$$

SISTEM

rješenjem sistema dobijemo stacionarne tačke koje mogu ali i ne moraju biti ekstrem

npr. $M(p_1, p_2)$ je jedna stacionarna tačka.

$$A = \frac{\partial^2 z(p_1, p_2)}{\partial x^2}$$

$$D = AC - B^2$$

$D > 0$ f-ja ima ekstrem u tački $M(p_1, p_2)$

a) $A > 0$ imamo Z_{\min}

b) $A < 0$ imamo Z_{\max}

$$C = \frac{\partial^2 z(p_1, p_2)}{\partial y^2}$$

$D < 0$ f-ja nema ekstrem

$D = 0$ potrebno ispitati ponašanje f-je u okolini stacionarne tačke:

$$\Delta z(M) = z(p_1 + \varepsilon, p_2 + \omega) - z(p_1, p_2) \quad \text{-privaškej f-je}$$

$\Delta z \geq 0 \quad \forall \varepsilon; \forall \omega \Rightarrow$ u tački M f-ja ima minimum

$\Delta z \leq 0 \quad \forall \varepsilon; \forall \omega \Rightarrow$ u tački M f-ja ima maksimum

⊕) Nađi ekstrema f-je $z = x^2 - 2x - y - \ln(2-y) + 4$.

kj. $D: 2-y > 0$

$$\frac{\partial z}{\partial x} = 2x - 2$$

$$2x - 2 = 0$$

$$\frac{\partial z}{\partial y} = -1 - \frac{1}{2-y} \cdot (-1) = \frac{1}{2-y} - 1$$

$$\frac{1}{2-y} - 1 = 0$$

$$x = 1, y = 1$$

Tačka $M(1,1)$ je stacionarna tačka
(kandidat za ekstrem)

$$(2-y)^{-1}$$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$M(1,1)$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0$$

$$A = 2, B = 0, C = 1$$

$$D = AC - B^2 = 2 > 0$$

$$\frac{\partial^2 z}{\partial y^2} = (-1)(2-y)^{-2} \cdot (-1) = \frac{1}{(2-y)^2}$$

F-ja ima ekstrem.

$A > 0 \Rightarrow$ f-ja ima minimum

$$z_{\min}(1,1) = 1 - 2 - 1 - \ln 1 + 4 = -2 + 4 = 2$$

Ⓝ Nadi ekstreme f-je $z = x^3 - 5xy + 5y^2 + 7x - 15y$.

R. j) Pronađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = 3x^2 - 5y + 7$$

$$3x^2 - 5y + 7 = 0 \quad | \cdot 2$$

$$-5x + 10y - 15 = 0$$

$$6x^2 - 5x - 1 = 0$$

$$D = 25 + 24 = 49$$

$$x_{1,2} = \frac{5 \pm 7}{2 \cdot 6}$$

$$x_1 = \frac{-2}{2 \cdot 6} = -\frac{1}{6}, \quad x_2 = \frac{12}{12} = 1$$

$$6(x + \frac{1}{6})(x - 1) = 0$$

$$\frac{\partial z}{\partial y} = 10y - 5x - 15$$

$$6x^2 - 10y + 14 = 0$$

$$-5x + 10y - 15 = 0 \quad +$$

$$\text{Za } x_1 = -\frac{1}{6} \Rightarrow -5 \cdot (-\frac{1}{6}) + 10y - 15 = 0$$

$$10y = 15 - \frac{5}{6}$$

$$10y = \frac{90 - 5}{6} = \frac{85}{6}$$

$$y = \frac{\frac{85}{6}}{\frac{10}{2}} = \frac{17}{12}$$

$$x_2 = 1 \Rightarrow$$

$$-5 + 10y - 15 = 0$$

$$10y = 20$$

$$y = 2$$

Stacionarne tačke su $(1, 2)$ i $(-\frac{1}{6}, \frac{17}{12})$

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

Za $M_1(1, 2)$

$$A = 6, B = -5, C = 10, D = AC - B^2 = 60 - 25 > 0$$

f-ja ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$$\frac{\partial^2 z}{\partial y^2} = 10$$

$$Z_{\min}(1, 2) = 1 - 10 + 20 + 7 - 30 = 8 + 10 - 30 = 8 - 20 = -12$$

Za $M_2(-\frac{1}{6}, \frac{17}{12})$

$$A = -1, B = -5, C = 10, D = AC - B^2 = -10 - 25 = -35$$

F-ja u ovoj tački nema ekstrem

#) Nađi ekstreme f-je $z = x + y - \frac{3}{2} \ln(x^2 + y^2 + 1)$.

R.) Izračunajmo prve parcijalne izvode

$$\frac{\partial z}{\partial x} = 1 - \frac{3}{2} \cdot \frac{1}{x^2 + y^2 + 1} \cdot 2x = 1 - \frac{3x}{x^2 + y^2 + 1}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{3}{2} \cdot \frac{1}{x^2 + y^2 + 1} \cdot 2y = 1 - \frac{3y}{x^2 + y^2 + 1}$$

Nađimo stacionarne tačke

$$1 - \frac{3x}{x^2 + y^2 + 1} = 0$$

$$1 - \frac{3x}{2x^2 + 1} = 0$$

$$/ \cdot 2x^2 + 1$$

$$1 - \frac{3y}{x^2 + y^2 + 1} = 0$$

$$2x^2 + 1 - 3x = 0$$

$$x_{1,2} = \frac{3 \pm 1}{4}$$

$$2x^2 - 3x + 1 = 0$$

$$x_1 = \frac{2}{4} = \frac{1}{2}$$

$$3x = 3y \Rightarrow x = y$$

$$D = 9 - 8 = 1$$

$$x_2 = 1$$

Stacionarne tačke su $M_1(\frac{1}{2}, \frac{1}{2})$ i $M_2(1, 1)$

Nađimo druge parcijalne izvode

$$\frac{\partial^2 z}{\partial x^2} = 0 - 3 \cdot \frac{1 \cdot (x^2 + y^2 + 1) - x \cdot 2x}{(x^2 + y^2 + 1)^2} = -3 \cdot \frac{-x^2 + y^2 + 1}{(x^2 + y^2 + 1)^2} = 3 \frac{x^2 - y^2 - 1}{(x^2 + y^2 + 1)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0 - 3x \cdot (-1) \cdot (x^2 + y^2 + 1)^{-2} \cdot 2y = 6 \frac{xy}{(x^2 + y^2 + 1)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = 0 - 3 \frac{1 \cdot (x^2 + y^2 + 1) - y \cdot 2y}{(x^2 + y^2 + 1)^2} = -3 \frac{x^2 - y^2 + 1}{(x^2 + y^2 + 1)^2}$$

$$\text{Za } M_1(\frac{1}{2}, \frac{1}{2}), A = 3 \cdot \frac{-1}{(\frac{1}{2} + 1)^2} = \frac{-3}{\frac{9}{4}} = \frac{-12}{9} = -\frac{4}{3}, B = \frac{2}{3}, C = -\frac{4}{3}$$

$$D = AC - B^2 = \frac{16}{9} - \frac{4}{9} > 0 \text{ f-ja ima ekstrem u tački } M_1$$

$$A < 0 \text{ f-ja ima minimum } z_{\min}(\frac{1}{2}, \frac{1}{2}) = 1 - \frac{3}{2} \ln \frac{3}{2}$$

$$\text{Za } M_2(1, 1), A = -\frac{1}{3}, B = \frac{2}{3}, C = -\frac{1}{3}$$

$$D = AC - B^2 = \frac{1}{9} - \frac{4}{9} < 0 \text{ f-ja u tački } M_2 \text{ nema ekstrem}$$

#) Nađi ekstreme f-je $z = x + y - \frac{3}{2} \ln(x^2 + y^2 + 1)$.

R) Pronađimo prve parcijalne izvode

$$\frac{\partial z}{\partial x} = 1 - \frac{3}{2} \cdot \frac{2x}{x^2 + y^2 + 1} = 1 - \frac{3x}{x^2 + y^2 + 1}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{3}{2} \cdot \frac{2y}{x^2 + y^2 + 1} = 1 - \frac{3y}{x^2 + y^2 + 1}$$

Pronađimo stacionarne tačke

$$\left. \begin{aligned} \frac{\partial z}{\partial x} = 0 &\Rightarrow 1 = \frac{3x}{x^2 + y^2 + 1} \\ \frac{\partial z}{\partial y} = 0 &\Rightarrow 1 = \frac{3y}{x^2 + y^2 + 1} \end{aligned} \right\} \Rightarrow x = y \text{ (deljenjem jednačina)}$$

Sad imamo $x = y$ i $1 = \frac{3x}{x^2 + y^2 + 1} \Rightarrow 1 = \frac{3x}{2x^2 + 1} \Rightarrow 2x^2 - 3x + 1 = 0$

$$D = 9 - 8 = 1 \\ x_1 = 1, x_2 = \frac{1}{2}$$

Stacionarne tačke su $M_1(1, 1)$ i $M_2(\frac{1}{2}, \frac{1}{2})$.

Pronađimo druge parcijalne izvode.

$$\frac{\partial^2 z}{\partial x^2} = \left(1 - \frac{3x}{x^2 + y^2 + 1}\right)'_x = \frac{-3(x^2 + y^2 + 1) + 3x \cdot 2x}{(x^2 + y^2 + 1)^2} = \frac{3x^2 - 3y^2 - 3}{(x^2 + y^2 + 1)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \left(1 - \frac{3y}{x^2 + y^2 + 1}\right)'_y = \left| \begin{array}{l} \text{zbog} \\ \text{simetričnosti} \end{array} \right| = \frac{3y^2 - 3x^2 - 3}{(x^2 + y^2 + 1)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{3x \cdot 2y}{(x^2 + y^2 + 1)^2} = \frac{6xy}{(x^2 + y^2 + 1)^2}$$

Za tačku $M_1(1, 1)$: $A = -\frac{3}{9} = -\frac{1}{3}$, $B = \frac{6}{9} = \frac{2}{3}$, $C = -\frac{3}{9} = -\frac{1}{3}$, $D = AC - B^2$
 $D = \frac{1}{9} - \frac{4}{9} < 0 \Rightarrow$ u M_1 f-ja nema ekstremu

Za tačku $M_2(\frac{1}{2}, \frac{1}{2})$: $A = \frac{-3}{(\frac{3}{2})^2} = -\frac{3}{\frac{9}{4}} = -\frac{12}{9} = -\frac{4}{3} \Rightarrow C = -\frac{4}{3}$
 $B = \frac{\frac{3}{2}}{\frac{9}{4}} = \frac{12}{18} = \frac{2}{3}$, $D = AC - B^2 = \frac{16}{9} - \frac{4}{9} = \frac{12}{9} = \frac{4}{3} > 0 \Rightarrow$ f-ja u tački M_2 ima ekstrem

$A < 0 \Rightarrow$ u M_2 f-ja ima maximum. $z_{\max}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} + \frac{1}{2} - 3 \ln(\frac{1}{4} + \frac{1}{4} + 1) = 1 - \ln \frac{27}{8}$

⊕) Nadi ekstremne f-je $z = \frac{8}{x} + \frac{x^2}{y} + y + 1$.

Rj. Pronađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = 8 \cdot (-1) x^{-2} + 2 \frac{x}{y} = \frac{-8}{x^2} + 2 \frac{x}{y}$$

$$\frac{\partial z}{\partial y} = x^2 \cdot (-1) y^{-2} + 1 = \frac{-x^2}{y^2} + 1$$

$$-\frac{8}{x^2} + \frac{2x}{y} = 0$$

$$-\frac{x^2}{y^2} + 1 = 0$$

$$\frac{8}{x^2} - 2 \frac{x}{y} = 0$$

$$\frac{x^2}{y^2} = 1 \Rightarrow \left(\frac{x}{y}\right)^2 = 1$$

Prva tome $\frac{x}{y} = 1$ i $\frac{x}{y} = -1$

Za $\frac{x}{y} = 1 \Rightarrow \frac{8}{x^2} - 2 \cdot 1 = 0$

$$\frac{8}{x^2} = 2 \quad | \cdot x^2 (x \neq 0)$$

$$2x^2 = 8$$

$$x^2 = 4$$

$$x_1 = -2, x_2 = 2$$

$$x_1 = -2 \Rightarrow \frac{x}{y} = 1$$

$$y = -2$$

$$(-2, -2)$$

Za $x_2 = 2 \Rightarrow$

$$\Rightarrow \frac{x}{y} = 1$$

$$y_2 = 2$$

$$(2, 2)$$

Za $\frac{x}{y} = -1$ imamo

$$\frac{8}{x^2} + 2 = 0$$

$$\frac{8}{x^2} = -2 \quad | \cdot x^2 (x \neq 0)$$

$$-2x^2 = 8$$

ova jednačina nema rešenja u skupu realnih brojeva

Stacionarne tačke su $M_1(-2, -2)$ i $M_2(2, 2)$.

Nadimo druge parcijalne izvode

$$\frac{\partial^2 z}{\partial x^2} = (-8)(-2)x^{-3} + \frac{2}{y} = \frac{16}{x^3} + \frac{2}{y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2x \cdot (-1) y^{-2} = \frac{-2x}{y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = -x^2 \cdot (-2) y^{-3} = \frac{2x^2}{y^3}$$

Za $M_1(-2, -2)$

$$A = \frac{16}{-8} + \frac{2}{-2} = -2 - 1 = -3$$

$$B = \frac{-2 \cdot (-2)}{4} = 1, \quad C = \frac{2 \cdot 4}{-8} = -1$$

$$D = AC - B^2 = 3 - 1 = 2 > 0$$

f-ja u tački $M_1(-2, -2)$ ima ekstrem.

A < 0 f-ja ima maksimum

$$Z_{\max}(-2, -2) = -4 - 2 - 2 + 1 = -7$$

Za $M_2(2, 2)$

$$A = 2 + 1 = 3, \quad B = \frac{-4}{4} = -1, \quad C = \frac{8}{8} = 1$$

$$D = AC - B^2 = 3 - 1 = 2 > 0 \quad \text{f-ja ima ekstrem}$$

A > 0 \Rightarrow f-ja ima minimum

$$Z_{\min}(2, 2) = 4 + 2 + 2 + 1 = 9$$

Ⓝ Naći ekstreme f-je $z = (x^2 + y) \sqrt{e^y}$.

$$f_j: \frac{\partial z}{\partial x} = 2x \sqrt{e^y}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= \sqrt{e^y} + (x^2 + y) \frac{1}{2\sqrt{e^y}} \cdot e^y = \sqrt{e^y} + (x^2 + y) \cdot \frac{1}{2} \sqrt{e^y} \\ &= \left(\frac{1}{2}x^2 + \frac{1}{2}y + 1\right) \sqrt{e^y} = \frac{1}{2}(x^2 + y + 2) \sqrt{e^y} \end{aligned}$$

$$2x \sqrt{e^y} = 0$$

$$\frac{1}{2}(x^2 + y + 2) \sqrt{e^y} = 0$$

$$e^y > 0 \quad \forall y \in \mathbb{R}$$

prema tome $x = 0$

$$\sqrt{e^y} > 0 \quad \forall y \in \mathbb{R}$$

$$x^2 + y + 2 = 0$$

$$x = 0 \Rightarrow y + 2 = 0$$

$$y = -2$$

$M(0, -2)$ je stacionarna tačka
(kandidat za ekstrem)

$$\frac{\partial^2 z}{\partial x^2} = 2\sqrt{e^y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 1x \frac{1}{2\sqrt{e^y}} \cdot e^y = x \sqrt{e^y}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{2} \sqrt{e^y} + \left(\frac{1}{2}x^2 + \frac{1}{2}y + 1\right) \frac{e^y \cdot \sqrt{e^y}}{2\sqrt{e^y} \cdot \sqrt{e^y}} = \frac{1}{2} \sqrt{e^y} \left(\frac{1}{2}x^2 + \frac{1}{2}y + 2\right)$$

$M(0, -2)$

$$A = 2 \cdot \sqrt{e^{-2}} = 2 \cdot \frac{1}{\sqrt{e^2}}$$

$$D = AC - B^2 = \frac{2}{\sqrt{e^2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{e^2}} = \frac{1}{e^2}$$

$$B = 0$$

$$C = \frac{1}{2} \sqrt{e^{-2}} \left(\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot (-2) + 2\right) = \frac{1}{2} \sqrt{\frac{1}{e^2}}$$

$D > 0 \Rightarrow$ f-ja ima
ekstrem

$A > 0 \Rightarrow$ f-ja ima
minimum

$$z_{\min}(0, -2) = (0 - 2) \sqrt{e^{-2}} = (-2) \cdot \frac{1}{\sqrt{e^2}} \approx -0.7358$$

Ⓝ) Nađi ekstreme f-je $z = e^{-2x^2}(x-y^2)$.

R) Nađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = e^{-2x^2} \cdot (-4)x(x-y^2) + e^{-2x^2} \cdot 1 = e^{-2x^2}(-4x^2 + 4xy^2 + 1)$$

$$\frac{\partial z}{\partial y} = e^{-2x^2} \cdot (-2)y = -2ye^{-2x^2}$$

$$e^{-2x^2}(-4x^2 + 4xy^2 + 1) = 0$$

e^{-2x^2} je uvijek pozitivno

$$-2ye^{-2x^2} = 0$$

$$-4x^2 + 4xy^2 + 1 = 0$$

$$-2y = 0 \Rightarrow y = 0$$

$$-4x^2 + 1 = 0$$

$$x^2 = \frac{1}{4} \Rightarrow x_1 = -\frac{1}{2}, x_2 = \frac{1}{2}$$

Stacionarne tačke

su $M_1(-\frac{1}{2}, 0)$ i

$M_2(\frac{1}{2}, 0)$

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= e^{-2x^2} \cdot (-4x)(-4x^2 + 4xy^2 + 1) + e^{-2x^2}(-8x + 4y^2) = \\ &= e^{-2x^2}(16x^3 - 16x^2y^2 - 4x - 8x + 4y^2) = e^{-2x^2}(16x^3 - 16x^2y^2 - 12x + 4y^2) \\ &= 4e^{-2x^2}(4x^3 - 4x^2y^2 - 3x + y^2) \end{aligned}$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^{-2x^2}(8xy) = 8xye^{-2x^2}$$

Za tačku $M_1(-\frac{1}{2}, 0)$

$$\begin{aligned} A &= 4e^{-2 \cdot \frac{1}{4}}(4 \cdot (-\frac{1}{8}) - 4 \cdot \frac{1}{4} \cdot 0 - 3 \cdot (-\frac{1}{2}) + 0) = 4e^{-\frac{1}{2}}(-\frac{1}{2} + \frac{3}{2}) = \frac{4}{\sqrt{e}} \\ B &= 0, C = -2e^{-2 \cdot \frac{1}{4}} = -2e^{-\frac{1}{2}} = \frac{-2}{\sqrt{e}} \end{aligned}$$

$$D = AC - B^2 = \frac{-8}{e} < 0$$

f-ja z u tački M_1 nema ekstrem

Za tačku $M_2(\frac{1}{2}, 0)$

$$\begin{aligned} A &= 4e^{-2 \cdot \frac{1}{4}}(4 \cdot \frac{1}{8} - 0 - 3 \cdot \frac{1}{2} + 0) = \\ &= 4e^{-\frac{1}{2}}(\frac{1}{2} - \frac{3}{2}) = \frac{-4}{\sqrt{e}} \end{aligned}$$

$$B = 0, C = -2e^{-2 \cdot \frac{1}{4}} = \frac{-2}{\sqrt{e}}$$

$$D = AC - B^2 = \frac{8}{e} > 0 \Rightarrow \text{f-ja za u tački } M_2 \text{ ima ekstrem}$$

$$A < 0 \Rightarrow z_{\max}(\frac{1}{2}, 0) = e^{-2 \cdot \frac{1}{4}}(\frac{1}{2} - 0) = \frac{1}{2} \cdot e^{-\frac{1}{2}} = \frac{1}{2\sqrt{e}}$$

Odrediti ekstremne vrijednosti f-je

$$z = \frac{x \cdot y}{2} + (47 - x - y) \left(\frac{x}{3} + \frac{y}{4} \right)$$

Rj.

$$\frac{\partial z}{\partial x} = \frac{1}{2}y + (-1) \left(\frac{x}{3} + \frac{y}{4} \right) + (47 - x - y) \cdot \frac{1}{3} = \frac{1}{2}y - \frac{1}{3}x - \frac{1}{4}y + \frac{47}{3} - \frac{1}{3}x - \frac{1}{3}y$$

$$= -\frac{2}{3}x + \frac{6-3-4}{12}y + \frac{47}{3} = -\frac{2}{3}x - \frac{1}{12}y + \frac{47}{3}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}x + (-1) \left(\frac{x}{3} + \frac{y}{4} \right) + (47 - x - y) \cdot \frac{1}{4} = \frac{1}{2}x - \frac{1}{3}x - \frac{1}{4}y + \frac{47}{4} - \frac{1}{4}x - \frac{1}{4}y$$

$$= -\frac{1}{12}x - \frac{1}{2}y + \frac{47}{4}$$

$$-\frac{2}{3}x - \frac{1}{12}y + \frac{47}{3} = 0 \quad | \cdot 12$$

$$-\frac{1}{12}x - \frac{1}{2}y + \frac{47}{4} = 0 \quad | \cdot 12$$

$$-8x - y + 188 = 0$$

$$-x - 6y + 141 = 0$$

$$-8x - y + 188 = 0$$

$$x = -6y + 141$$

$$-8(-6y + 141) - y + 188 = 0$$

$$48y - 1128 - y + 188 = 0$$

$$47y = 940$$

$$y = 20$$

$$x = -6y + 141 = -120 + 141 = 21$$

Stacionarna tačka je $M(21, 20)$.

$$\frac{\partial^2 z}{\partial x^2} = -\frac{2}{3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{12}$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{2}$$

$$D = AC - B^2$$

$$M(21, 20)$$

$$A = -\frac{2}{3}, \quad B = -\frac{1}{12}, \quad C = -\frac{1}{2}$$

$$D = \frac{2}{6} - \frac{1}{144} = \frac{1}{3} - \frac{1}{144} > 0$$

f-ja z ima ekstrem

$A < 0$ f-ja ima maksimum

$$Z_{\max}(21, 20) = 21 \cdot 20 + (47 - 41) \cdot (7 + 5) = 210 + 6 \cdot 12 = 210 + 72 = 282$$

$$Z_{\max}(21, 20) = 282 \quad \text{traženi ekstrem f-je}$$

○ Nadi ekstreme f-je $z = x^4 + y^4 - 2x^2$.

Rj: $\frac{\partial z}{\partial x} = 4x^3 - 4x$

$\frac{\partial z}{\partial y} = 4y^3$

$4x^3 - 4x = 0 \quad | :4$
 $4y^3 = 0 \quad | :4$

$x^3 - x = 0$
 $y^3 = 0$

$x(x^2 - 1) = 0$

$y^3 = 0$

$x(x-1)(x+1) = 0$

$y^3 = 0$

$y = 0 \wedge (x_1 = 0, x_2 = 1, x_3 = -1)$

Stacionarne tačke f-je su $M_1(-1, 0)$, $M_2(0, 0)$; $M_3(1, 0)$.

$\frac{\partial^2 z}{\partial x^2} = 12x^2 - 4$

za $M_1(-1, 0)$, $A = 8$, $B = 0$, $C = 0$

$\frac{\partial^2 z}{\partial x \partial y} = 0$

$D = 0$ ispitujemo ponašanje f-je u okolini tačke $M_1(-1, 0)$

$\frac{\partial^2 z}{\partial y^2} = 12y^2$

$\Delta z = z(-1+\epsilon, 0+\omega) - z(-1, 0) =$
 $= (-1+\epsilon)^4 + \omega^4 - 2(-1+\epsilon)^2 - [(-1)^4 + 0^4 - 2(-1)^2]$
 $= 1 - 4\epsilon + 6\epsilon^2 - 4\epsilon^3 + \epsilon^4 + \omega^4 - 2(1 - 2\epsilon + \epsilon^2) - (1 - 2)$
 $= 1 - 4\epsilon + 6\epsilon^2 - 4\epsilon^3 + \epsilon^4 + \omega^4 - 2 + 4\epsilon - 2\epsilon^2 + 1$
 $= \epsilon^4 - 4\epsilon^3 + 4\epsilon^2 + \omega^4 = \epsilon^2(\epsilon^2 - 4\epsilon + 4) + \omega^4$
 $= \epsilon^2(\epsilon - 2)^2 + \omega^4 \geq 0 \quad \forall \epsilon; \forall \omega$

paskulov trougao
 $\begin{matrix} & & 1 & & \\ & 1 & & 1 & \\ & 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{matrix}$

f-ja ima minimum u tački $M_1(-1, 0)$, $z_{min} = -1$

za $M_2(0, 0)$, $A = -4$, $B = 0$, $C = 0$, $D = AC - B^2 = 0$

ispitujemo ponašanje f-je u okolini tačke

$\Delta z = z(0+\epsilon, 0+\omega) - z(0, 0) = \epsilon^4 + \omega^4 - 2\epsilon^2 = \epsilon^2(\epsilon^2 - 2) + \omega^4$

$\epsilon = 0: \Delta z = \omega^4$

$\omega = 0: \Delta z = \epsilon^2(\epsilon^2 - 2) \Rightarrow \Delta z < 0 \text{ za } \epsilon^2 < 2$
 $\Delta z > 0 \text{ za } \epsilon^2 > 2$

u tački M_2

Privađtej f-je je promjenjivog znaka pa f-ja nema ekstrem!

za $M_3(1, 0)$, $A = 8$, $B = 0$, $C = 0$, $D = AC - B^2 = 0$ ispitujemo ponašanje f-je u okolini tačke

$\Delta z = z(1+\epsilon, 0+\omega) - z(1, 0) = (1+\epsilon)^4 + \omega^4 - 2(1+\epsilon)^2 - (1 - 2)$
 $= 1 + 4\epsilon + 6\epsilon^2 + 4\epsilon^3 + \epsilon^4 + \omega^4 - 2 - 4\epsilon - 2\epsilon^2 + 1 = \epsilon^4 + 4\epsilon^3 + 4\epsilon^2 + \omega^4$

$\Delta z = \epsilon^2(\epsilon + 2)^2 + \omega^4 \geq 0 \quad \forall \epsilon; \forall \omega$ f-ja z u tački M_3 ima min
 $z_{min} = -1$

#) Nađi ekstreme f-je $z = (2x^2 + 3y^2) e^{-(x^2 + y^2)}$

Rj.

$$\frac{\partial z}{\partial x} = 4x \cdot e^{-x^2-y^2} + (2x^2+3y^2) e^{-x^2-y^2} \cdot (-2x) = (4x - 4x^3 - 6xy^2) e^{-x^2-y^2}$$

$$\frac{\partial z}{\partial y} = 6y e^{-x^2-y^2} + (2x^2+3y^2) e^{-x^2-y^2} \cdot (-2y) = (6y - 4x^2y - 6y^3) e^{-x^2-y^2}$$

$$2x(2 - 2x^2 - 3y^2) e^{-x^2-y^2} = 0$$

$$e^{-x^2-y^2} \neq 0 \quad \forall (x, y \in \mathbb{R})$$

$$2y(3 - 2x^2 - 3y^2) e^{-x^2-y^2} = 0$$

$$\text{ili } 2 - 2x^2 - 3y^2 = 0 \quad ; \quad y = 0$$

$$2x^2 = 2 \quad M_4(-1, 0)$$

$$x^2 = 1$$

$$x_{1,2} = \pm 1$$

$$M_5(1, 0)$$

$$x=0 \quad ; \quad y=0, \quad M_1(0, 0)$$

ili

$$x=0 \quad ; \quad 3 - 2x^2 - 3y^2 = 0$$

$$M_2(0, -1) \quad 3y^2 = 3$$

$$y^2 = 1$$

$$M_3(0, 1) \quad y_{1,2} = \pm 1$$

ili

$$2 - 2x^2 - 3y^2 = 0$$

$$- 3 - 2x^2 - 3y^2 = 0$$

$$\hline -1 = 0$$

sistem
nema
rešenja

Stacionarne tačke su M_1, M_2, M_3, M_4 i M_5 .

$$\frac{\partial^2 z}{\partial x^2} = (4 - 12x^2 - 6y^2) e^{-x^2-y^2} + (4x - 4x^3 - 6xy^2) e^{-x^2-y^2} (-2x) = (8x^4 + 12x^2y^2 - 20x^2 - 6y^2 + 4) e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = (-12xy) e^{-x^2-y^2} + (4x - 4x^3 - 6xy^2) e^{-x^2-y^2} (-2y) = (-20xy + 8x^3y + 12xy^3) e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = (6 - 4x^2 - 18y^2) e^{-x^2-y^2} + (6y - 4x^2y - 6y^3) e^{-x^2-y^2} (-2y) = (-30y^2 + 12y^4 + 8x^2y^2 - 4x^2 + 6) e^{-x^2-y^2}$$

za $M_1(0,0)$, $A=4$, $B=0$, $C=6$, $D=AC-B^2=24 > 0$ ima ekstrem

$A > 0$ ima minimum, $Z_{\min}(0,0) = 0$

za $M_2(0,-1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2} > 0$ ima ekstrem

$A < 0$ ima maksimum, $Z_{\max}(0,-1) = 3e^{-1}$

za $M_3(0,1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2} > 0$ ima ekstrem

$A < 0$ ima maksimum $Z_{\max}(0,1) = 3e^{-1}$

za $M_4(-1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$, $D=AC-B^2=-16e^{-2} < 0$

f-ja u tački $M_4(-1,0)$ nema ekstrem

za $M_5(1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$

f-ja u tački $M_5(1,0)$ nema ekstrem

Naći stacionarne tačke f-je $z = xy \ln(x^2 + y^2)$.

Rj.

$$\frac{\partial z}{\partial x} = y \ln(x^2 + y^2) + xy \cdot \frac{1}{x^2 + y^2} \cdot 2x = y \ln(x^2 + y^2) + \frac{2x^2 y}{x^2 + y^2}$$

$$\frac{\partial z}{\partial y} = x \ln(x^2 + y^2) + xy \cdot \frac{1}{x^2 + y^2} \cdot 2y = x \ln(x^2 + y^2) + \frac{2xy^2}{x^2 + y^2}$$

$$y \ln(x^2 + y^2) + \frac{2x^2 y}{x^2 + y^2} = 0$$

$$x \ln(x^2 + y^2) + \frac{2xy^2}{x^2 + y^2} = 0$$

$$y \left(\ln(x^2 + y^2) + \frac{2x^2}{x^2 + y^2} \right) = 0$$

$$x \left(\ln(x^2 + y^2) + \frac{2y^2}{x^2 + y^2} \right) = 0$$

$$y=0 \quad \text{ili} \quad \ln(x^2 + y^2) + \frac{2x^2}{x^2 + y^2} = 0$$

$$x=0 \quad \text{ili} \quad \ln(x^2 + y^2) + \frac{2y^2}{x^2 + y^2} = 0$$

$$\text{ili} \quad \ln(x^2 + y^2) + \frac{2x^2}{x^2 + y^2} = 0 \quad (1)$$

$$\ln(x^2 + y^2) + \frac{2y^2}{x^2 + y^2} = 0 \quad (2)$$

$$(1) - (2): \quad \frac{2x^2}{x^2 + y^2} - \frac{2y^2}{x^2 + y^2} = 0$$

$$2x^2 - 2y^2 = 0$$

$$\text{za } y = -x: \quad \ln(2x^2) + 1 = 0$$

$$x_{1,2} = \pm \frac{1}{\sqrt{2e}}$$

$$M_8 \left(-\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}} \right)$$

$$M_9 \left(\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}} \right)$$

$$\text{za } x=y: \quad \ln(2x^2) + 1 = 0$$

$$\ln(2x^2) = -1$$

$$e^{-1} = 2x^2$$

$$x^2 = \frac{1}{2e}$$

$$x_{1,2} = \pm \frac{1}{\sqrt{2e}}$$

$$M_6 \left(-\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}} \right), M_7 \left(\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}} \right)$$

Stacionarne tačke su: $M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9$.

$y=0$; $x=0$ $M_1(0,0)$
 ili
 za M_1 f-ja nije definisana

$$y=0 ; \quad \ln(x^2 + y^2) + \frac{2y^2}{x^2 + y^2} = 0$$

$$\ln x^2 = 0$$

$$x^2 = 1$$

$$x_1 = -1, x_2 = 1$$

$$M_2(-1,0), M_3(1,0)$$

ili

$$\ln(x^2 + y^2) + \frac{2x^2}{x^2 + y^2} = 0 \quad ; \quad x=0$$

$$\ln y^2 = 0$$

$$M_4(0,-1)$$

$$y_1 = -1, y_2 = 1$$

$$M_5(0,1)$$

$$2(x^2 - y^2) = 0$$

$$2(x-y)(x+y) = 0$$

$$x=y \quad \text{ili} \quad x=-y$$

Uslovni ekstremi f-je dviju promjenjivi

Ako trebamo naći ekstrem f-je $z=f(x,y)$ tako da x i y zadovoljavaju neki uslov $g(x,y)=0$ tada tražimo ekstrem Lagranžove f-je $F(x,y,\lambda)=f(x,y)+\lambda g(x,y)$.

$$\frac{\partial F}{\partial x} = 0$$

$$\frac{\partial F}{\partial y} = 0$$

$$\frac{\partial F}{\partial \lambda} = 0$$

SISTEM

rješavanjem sistema dobijemo neke stacionarne tačke i dalji proces se nastavlja kao kod traženja ekstrema f-je dvije promjenjive

|| način: neka je $M(p_1, p_2)$ neka stacionarna tačka

$$d^2F(p_1, p_2) = F''_{xx}(p_1, p_2) dx^2 + 2F''_{xy}(p_1, p_2) dx dy + F''_{yy}(p_1, p_2) dy^2$$

$$d^2F(p_1, p_2) > 0 \Rightarrow z_{\min}(p_1, p_2)$$

$$d^2F(p_1, p_2) < 0 \Rightarrow z_{\max}(p_1, p_2)$$

Ako se desi slučaj da imamo više uslova, onda uvodimo više parametara (λ, μ, \dots) .

1. Naći ekstreme f-je $z=6-4x-3y$ uz uslov $x^2+y^2=1$.

Rj. $F(x,y) = 6-4x-3y + \lambda(x^2+y^2-1) \neq \emptyset$

$$\frac{\partial F}{\partial x} = -4 + 2\lambda x$$

$$\frac{\partial F}{\partial y} = -3 + 2\lambda y$$

$$\frac{\partial F}{\partial \lambda} = x^2 + y^2 - 1$$

$$2\lambda x - 4 = 0$$

$$2\lambda y - 3 = 0$$

$$x^2 + y^2 - 1 = 0$$

$$2\lambda x = 4$$

$$2\lambda y = 3$$

$$x^2 + y^2 = 1$$

$$x = \frac{2}{\lambda}$$

$$y = \frac{3}{2\lambda}$$

$$x^2 + y^2 = 1$$

$$\frac{4}{\lambda^2} + \frac{9}{4\lambda^2} = 1$$

$$\frac{25}{4\lambda^2} = 1$$

$$4\lambda^2 = 25$$

$$\lambda_{1,2} = \pm \frac{5}{2}$$

$$\lambda_1 = -\frac{5}{2} \Rightarrow x_1 = -\frac{4}{5}; y_1 = \frac{3}{2 \cdot (-\frac{5}{2})} = -\frac{3}{5}$$

$$\lambda_2 = \frac{5}{2} \Rightarrow x_2 = \frac{2}{\frac{5}{2}} = \frac{4}{5}; y_2 = \frac{3}{2 \cdot \frac{5}{2}} = \frac{3}{5}$$

Stacionarne tačke

su $M(-\frac{4}{5}, -\frac{3}{5})$ za $\lambda = -\frac{5}{2}$

i $N(\frac{4}{5}, \frac{3}{5})$ za $\lambda = \frac{5}{2}$.

$$\frac{\partial^2 F}{\partial x^2} = 2\lambda$$

$$\text{za } M(-\frac{4}{5}, -\frac{3}{5}), \lambda = -\frac{5}{2}$$

$$A = -5, B = 0, C = -5, D = AC - B^2 = 25 > 0$$

f-ja ima ekstrem, $A < 0$ f-ja ima maksimum

$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

$$\frac{\partial^2 F}{\partial y^2} = 2\lambda$$

$$Z_{\max}(-\frac{4}{5}, -\frac{3}{5}) = 6 - 4(-\frac{4}{5}) - 3(-\frac{3}{5}) = \frac{30 + 16 + 9}{5} = \frac{55}{5} = 11$$

$$\text{za } N(\frac{4}{5}, \frac{3}{5}), \lambda = \frac{5}{2}, A = 5, B = 0, C = 5, D = AC - B^2 = 25 > 0$$

f-ja ima ekstrem u tački N, $A > 0$ f-ja ima minimum

$$Z_{\min}(\frac{4}{5}, \frac{3}{5}) = 6 - 4 \cdot \frac{4}{5} - 3 \cdot \frac{3}{5} = \frac{30 - 16 - 9}{5} = \frac{5}{5} = 1$$

2) Naći uslovne ekstreme f-je $z = y + 2x + 3$ uz uslov $x^2 - 6x + y + 5 = 0$.

$$R_j: F(x, y) = 2x + y + 3 + \lambda(x^2 - 6x + y + 5)$$

$$-x = -3 - 1$$

$$\frac{\partial F}{\partial x} = 2 + 2\lambda x - 6\lambda$$

$$2\lambda x - 6\lambda + 2 = 0 \quad | :2$$

$$x = 4$$

$$\lambda + 1 = 0$$

$$x^2 - 6x + y + 5 = 0$$

$$\frac{\partial F}{\partial y} = 1 + \lambda$$

$$x^2 - 6x + y + 5 = 0$$

$$16 - 24 + y + 5 = 0$$

$$\frac{\partial F}{\partial \lambda} = x^2 - 6x + y + 5$$

$$\lambda x = 3\lambda - 1$$

$$y = 3$$

$$\lambda = -1$$

$$x^2 - 6x + y + 5 = 0$$

Tačka $M(4, 3)$ je stacionarna tačka, za $\lambda = -1$

$$\frac{\partial^2 F}{\partial x^2} = 2\lambda$$

$$M(4, 3), \lambda = -1$$

$$A = -2, B = 0, C = 0 \Rightarrow D = AC - B^2 = 0$$

$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

$$d^2 F = \frac{\partial^2 F}{\partial x^2} dx^2 + \frac{\partial^2 F}{\partial x \partial y} dx dy + \frac{\partial^2 F}{\partial y^2} dy^2$$

$$\frac{\partial^2 F}{\partial y^2} = 0$$

$$d^2 F = 2\lambda dx^2 \Rightarrow d^2 F = -2 dx^2 < 0$$

U tački $M(4, 3)$ f-ja ima maksimum, $Z_{\max}(4, 3) = 3 + 8 + 3 = 14$

3) Odrediti ekstreme f-je $z = x^2 + y^2$ uz uslov $\frac{x}{2} + \frac{y}{3} = 1$.

$$R_j: Z_{\min}(\frac{18}{13}, \frac{12}{13}) = \frac{36}{13}, \lambda = -\frac{72}{13}$$

4) Naći uslovne ekstreme f-je $z = \ln(x+y)$, ako je $x^2 + 2y^2 = 4$.

$$R_j: Z_{\max}(2\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}) = \ln(3\sqrt{\frac{2}{3}}), \lambda = -\frac{1}{8}$$

#) Nadi uslovne ekstreme f-je $z = 2x^4 + 8y^4 + 24$ ako je $8x + 4y = 1$.

Rj. $F(x, y, \lambda) = 2x^4 + 8y^4 + 24 + \lambda(8x + 4y - 1)$

$$8x + 4y - 1 = 0$$

$$8 \cdot 2y + 4y - 1 = 0$$

$$20y = 1$$

$$y = \frac{1}{20}$$

$$x = 2 \cdot \frac{1}{20} = \frac{1}{10}$$

$M_1(\frac{1}{10}, \frac{1}{20})$ je stacionarna tačka

$$\frac{\partial F}{\partial x} = 8x^3 + 8\lambda$$

$$8x^3 + 8\lambda = 0 \quad | :8$$

$$\frac{\partial F}{\partial y} = 32y^3 + 4\lambda$$

$$32y^3 + 4\lambda = 0 \quad | :4$$

$$\frac{\partial F}{\partial \lambda} = 8x + 4y - 1$$

$$x^3 + \lambda = 0$$

$$8y^3 + \lambda = 0$$

$$x^3 - 8y^3 = 0$$

$$x^3 = 8y^3$$

$$x = 2y$$

$$\frac{\partial^2 F}{\partial x^2} = 24x^2$$

$$D = AC - B^2$$

$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

$$M_1(\frac{1}{10}, \frac{1}{20})$$

$$A = 24 \cdot \frac{1}{100} = \frac{24}{100} = \frac{12}{50} = \frac{6}{25}$$

$$\frac{\partial^2 F}{\partial y^2} = 96y^2$$

$$B = 0$$

$$C = \frac{48}{20 \cdot 20} \cdot \frac{1}{10 \cdot 10} = \frac{24}{100} = \frac{12}{50} = \frac{6}{25}$$

$$D = (\frac{6}{25})^2 > 0 \quad f\text{-ja ima ekstrem}$$

$A > 0$ f-ja ima minimum

$$Z_{\min}(\frac{1}{10}, \frac{1}{20}) = 2 \cdot \frac{1}{10^4} + 8 \cdot \frac{1}{20^4} + 24 = \frac{2}{10000} + \frac{1}{20000} + 24 = \frac{2}{10000} + \frac{1}{20000} + \frac{24000}{1000} = \frac{4 + 1 + 480000}{20000} =$$

$$= \frac{480005}{20000} = \frac{96001}{4000}$$

$Z_{\min} = \frac{96001}{4000}$ je minimum f-je u tački $M(\frac{1}{10}, \frac{1}{20})$

#) Nađi uslovne ekstreme f-je $z = (x-y)^4 + 1$ ako je $x^2 + y^2 = 18$.

Rj. $F(x, y, \lambda) = (x-y)^4 + 1 + \lambda(x^2 + y^2 - 18)$

$$\frac{\partial F}{\partial x} = 4(x-y)^3 + 2\lambda x$$

$$4(x-y)^3 + 2\lambda x = 0 \quad \dots (1)$$

$$\frac{\partial F}{\partial y} = 4(x-y)^3 \cdot (-1) + 2\lambda y$$

$$-4(x-y)^3 + 2\lambda y = 0 \quad \dots (2)$$

$$\frac{\partial F}{\partial \lambda} = x^2 + y^2 - 18$$

$$x^2 + y^2 - 18 = 0 \quad \dots (3)$$

$$(1) - (2): 2\lambda x + 2\lambda y = 0 \quad | :2$$

$$\lambda(x+y) = 0$$

$$\lambda = 0 \quad \text{ili} \quad x+y = 0$$

a) $x+y = 0$

$$x = -y$$

$$(-y)^2 + y^2 = 18$$

$$2y^2 = 18$$

$$y_{1,2} = \pm 3$$

$$y_1 = -3 \Rightarrow x_1 = 3$$

$$y_2 = 3 \Rightarrow x_2 = -3$$

$$M_1(3, -3), M_2(-3, 3)$$

za $M_1(1) \Rightarrow 6\lambda = -4 \cdot 6^3$

za $M_2(1) \Rightarrow -6\lambda = -4 \cdot (-6)^3$

$$\Rightarrow \lambda = -144$$

b) $\lambda = 0$

$$(1) \Rightarrow 4(x-y)^3 = 0$$

$$x = y$$

$$2y^2 = 18$$

$$y_{3,4} = \pm 3$$

$$M_3(-3, -3)$$

$$M_4(3, 3)$$

$$\lambda = 0$$

Stacionarne tačke su $M_1(3, -3)$

$M_2(-3, 3)$ za $\lambda = -144$; $M_3(-3, -3)$

; $M_4(3, 3)$ za $\lambda = 0$.

$$\frac{\partial^2 F}{\partial x^2} = 12(x-y)^2 + 2\lambda$$

$$\frac{\partial^2 F}{\partial x \partial y} = -12(x-y)^2$$

$$\frac{\partial^2 F}{\partial y^2} = 12(x-y)^2 + 2\lambda$$

$$D = AC - B^2$$

$$M_1(3, -3), \lambda = -144$$

$$A = 12 \cdot 36 - 2 \cdot 144 = 144$$

$$B = -12 \cdot 36 = -432$$

$$C = 144$$

$$D = 20736 - 186624$$

f-ja u tački M_1 nema ekstrema

$$M_2(-3, 3), \lambda = -144$$

$$\left. \begin{array}{l} A = 144 \\ B = 432 \\ C = 144 \end{array} \right\} D < 0$$

f-ja u tački M_2 nema ekstrem

$$M_3(-3, -3), \lambda = 0$$

$A=0, B=0, C=0 \Rightarrow D=0$ potrebno je ispitati f-ju u okolini tačke $M_3(-3, -3)$

$$\Delta z(M_3) = z(-3+\epsilon, -3+\omega) - z(-3, -3) = (-3+\epsilon+3-\omega)^4 + 1 - 1 = (\epsilon-\omega)^4 > 0$$

Privaštaj; f-je u okolini tačke M_3 je pozitivna

pa f-ja u M_3 ima minimum, $z_{\min}(-3, -3) = 1$

$$M_4(3, 3), \lambda = 0$$

$A=0, B=0, C=0 \Rightarrow D=0$ potrebno je ispitati f-ju u okolini tačke $M_4(3, 3)$

$$\Delta z(M_4) = z(3+\epsilon, 3+\omega) - z(3, 3) = (3+\epsilon-3-\omega)^4 + 1 - 1 = (\epsilon-\omega)^4 > 0 \quad \forall \epsilon; \forall \omega$$

Privaštaj; f-je u okolini tačke M_4 je pozitivna

pa f-ja u M_4 ima minimum, $z_{\min}(3, 3) = 1$.

(#) Nadi uslovne ekstreme f-je $z = 2x + 4y$ ako je

$$\frac{2}{x} + \frac{4}{y} = 3.$$

Rj: Formirajmo Lagranžovu f-ju $F(x, y, \lambda) = 2x + 4y + \lambda(\frac{2}{x} + \frac{4}{y} - 3)$.

$$\frac{\partial F}{\partial x} = 2 + 2\lambda \cdot \frac{(-1)}{x^2} \quad \left[\left(\frac{1}{x}\right)' = (x^{-1})' = (-1)(x^{-2}) \right] \quad \left[(x^{-2})' = (-2)x^{-3} = \frac{-2}{x^3} \right]$$

$$\frac{\partial F}{\partial y} = 4 + 4\lambda \cdot \frac{(-1)}{y^2}$$

$$\frac{\partial F}{\partial \lambda} = \frac{2}{x} + \frac{4}{y} - 3$$

Formirajmo sistem

$$4 - \frac{4\lambda}{y^2} = 0 \quad | :4$$

$$2 - \frac{2\lambda}{x^2} = 0 \quad | :2$$

$$\frac{2}{x} + \frac{4}{y} = 3$$

$$1 - \frac{\lambda}{x^2} = 0$$

$$1 = \frac{\lambda}{x^2} \quad (1)$$

$$1 - \frac{\lambda}{y^2} = 0$$

$$1 = \frac{\lambda}{y^2} \quad (2)$$

$$(1) : (2) \Rightarrow \frac{\lambda}{x^2} = \frac{\lambda}{y^2} \Rightarrow x^2 = y^2$$

$$\frac{2}{x} + \frac{4}{y} = 3$$

$$\frac{2}{x} + \frac{4}{y} = 3 \quad (3)$$

$$tj: x = \pm y$$

Za $x = y$ iz (3) $\frac{2}{x} + \frac{4}{x} = 3$

$$\frac{6}{x} = 3 \Rightarrow x = 2 \Rightarrow y = 2$$

Za $x = -y$ iz (3)

$$\frac{2}{x} - \frac{4}{x} = 3 \Rightarrow -\frac{2}{x} = 3$$

$$3x = -2 \Rightarrow x = -\frac{2}{3}$$

$$\Rightarrow y = \frac{2}{3}$$

Za $M_1(2, 2) \Rightarrow 2 - 2\lambda \cdot \frac{1}{4} = 0$

$$\lambda = 4$$

Za $M_2(-\frac{2}{3}, \frac{2}{3}) \Rightarrow 2 - 2\lambda \cdot \frac{9}{4} = 0 \Rightarrow \lambda = \frac{4}{9}$

Stacionarne tačke su $M_1(2, 2)$ za $\lambda = 4$; $M_2(-\frac{2}{3}, \frac{2}{3})$ za $\lambda = \frac{4}{9}$.

$$\frac{\partial^2 F}{\partial x^2} = \frac{4\lambda}{x^3}$$

Za $M_1(2, 2), \lambda = 4$

$$A = \frac{16}{8} = 2, B = 0, C = \frac{32}{8} = 4, D = AC - B^2 = 8 > 0 \text{ f-ja ima ekstrem}$$

$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

$$A > 0 \Rightarrow \text{f-ja ima minimum}$$

$$z_{\min}(2, 2) = 4 + 8 = 12$$

Za $M_2(-\frac{2}{3}, \frac{2}{3}), \lambda = \frac{4}{9}$, $A = \frac{\frac{16}{9}}{-\frac{8}{27}} = -\frac{16 \cdot 27}{8 \cdot 9} = -2 \cdot 3 = -6$

$$B = 0, C = \frac{\frac{32}{9}}{\frac{8}{27}} = \frac{32 \cdot 27}{8 \cdot 9} = 4 \cdot 3 = 12, D = AC - B^2 = -72 < 0 \Rightarrow$$

\Rightarrow f-ja u tački M_2 nema ekstremnu vrijednost

(#) Nadi uslovna ekstreme f-je $z=xy$ ako je
 $x^2+y^2=2ax, a>0.$

Rj: Posmatramo f-ju $F(x,y,\lambda) = xy + \lambda(x^2+y^2-2ax)$

$$\frac{\partial F}{\partial x} = y + 2\lambda x - 2a\lambda = 0$$

$$y + 2\lambda x - 2a\lambda = 0$$

$$\frac{\partial F}{\partial y} = x + 2\lambda y = 0$$

$$x + 2\lambda y = 0$$

$$\frac{\partial F}{\partial \lambda} = x^2 + y^2 - 2ax = 0$$

$$x^2 + y^2 - 2ax = 0$$

$$(1) \quad y + 2\lambda(x-a) = 0 \Rightarrow x-a = \frac{-y}{2\lambda} \dots (1)$$

$$(2) \quad x = -2\lambda y$$

$$(3) \quad x^2 - 2x \cdot a + a^2 - a^2 + y^2 = 0$$

$$(2) \text{ u } (1): y + 2\lambda(-2\lambda y - a) = 0$$

$$(3): (x-a)^2 + y^2 = a^2$$

$$y - 4\lambda^2 y - 2a\lambda = 0$$

$$y(1 - 4\lambda^2) = 2a\lambda$$

$$y = \frac{2a\lambda}{1 - 4\lambda^2}$$

$$y = \frac{2a\lambda}{1 - 4\lambda^2} = \frac{2a\lambda}{\pm\sqrt{1+4\lambda^2}} \Rightarrow 1 - 4\lambda^2 = \pm\sqrt{1+4\lambda^2}$$

$$(1 - 4\lambda^2)^2 = 1 + 4\lambda^2$$

$$16\lambda^4 - 8\lambda^2 + 1 = 1 + 4\lambda^2$$

$$16\lambda^4 - 12\lambda^2 = 0$$

$$\lambda^2(16\lambda^2 - 12) = 0$$

$$\lambda_1 = 0$$

$$\lambda_{2,3} = \pm\sqrt{\frac{12}{16}} = \pm\sqrt{\frac{3}{4}}$$

$$= \pm\frac{\sqrt{3}}{2}$$

$$\lambda_1 = 0: \quad y = 0 \\ x = 0$$

$$\lambda_2 = \frac{\sqrt{3}}{2}: \quad y + \sqrt{3}x - a\sqrt{3} = 0$$

$$x + y\sqrt{3} = 0$$

$$x^2 + y^2 - 2ax = 0$$

$$\sqrt{3}x + y = a\sqrt{3}$$

$$-\sqrt{3}x + 3y = 0$$

$$-2y = a\sqrt{3}$$

$$x = -\frac{3}{2}a$$

$$y = -\frac{a}{2}\sqrt{3}$$

$$\lambda_3 = -\frac{\sqrt{3}}{2}: \quad y - x\sqrt{3} + a\sqrt{3} = 0$$

$$x - y\sqrt{3} = 0$$

$$x^2 + y^2 - 2ax = 0$$

$$-x\sqrt{3} + y = -a\sqrt{3}$$

$$+ x\sqrt{3} - 3y = 0$$

$$-2y = -a\sqrt{3}$$

$$y = \frac{\sqrt{3}}{2}a \Rightarrow x = \frac{3}{2}a$$

Stacionarne tačke su $M_1(0,0)$ za $\lambda=0$, $M_2(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a)$ za $\lambda=\frac{\sqrt{3}}{2}$; $M_3(\frac{3}{2}a, \frac{\sqrt{3}}{2}a)$ za $\lambda=-\frac{\sqrt{3}}{2}$.

$$\frac{\partial^2 F}{\partial x^2} = 2\lambda$$

$$M_1(0,0), \lambda=0$$

$D=AC-B^2=-1<0 \Rightarrow$ f-ja u tački $M_1(0,0)$ nema ekstrem

$$\frac{\partial^2 F}{\partial x \partial y} = 1$$

$$M_2(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a), \lambda=\frac{\sqrt{3}}{2}$$

$D=AC-B^2=3-1=2>0 \Rightarrow$ f-ja u tački M_2 ima ekstre

$$\frac{\partial^2 F}{\partial y^2} = 2\lambda$$

$A=\sqrt{3}>0 \Rightarrow$ f-ja ima minimum

$$Z_{\min}(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a) = -\frac{3\sqrt{3}}{4}a^2$$

$$M_3(\frac{3}{2}a, \frac{\sqrt{3}}{2}a) \text{ za } \lambda=-\frac{\sqrt{3}}{2}$$

$D=AC-B^2=3-1>0 \Rightarrow$ f-ja ima ekstrem

$A=-\sqrt{3}<0 \Rightarrow$ f-ja u tački M_3 ima maksimum

$$Z_{\max}(\frac{3}{2}a, \frac{\sqrt{3}}{2}a) = \frac{3\sqrt{3}}{4}a^2$$

Dio tablice integrala

$$1. \int u^a du = \frac{u^{a+1}}{a+1} + C, \quad a \neq -1.$$

$$2. \int u^{-1} du = \int \frac{du}{u} = \int \frac{u'}{u} dx = \ln|u| + C.$$

$$3. \int a^u du = \frac{a^u}{\ln a} + C; \int e^u du = e^u + C.$$

$$4. \int \sin u du = -\cos u + C.$$

$$5. \int \cos u du = \sin u + C.$$

$$6. \int \sec^2 u du = \operatorname{tg} u + C.$$

$$7. \int \operatorname{cosec}^2 u du = -\operatorname{ctg} u + C.$$

$$8. \int \frac{du}{u^2+a^2} = \frac{1}{a} \operatorname{arc} \operatorname{tg} \frac{u}{a} + C.$$

$$9. \int \frac{du}{u^2-a^2} = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right| + C.$$

$$10. \int \frac{du}{\sqrt{a^2-u^2}} = \operatorname{arc} \sin \frac{u}{a} + C.$$

$$11. \int \frac{du}{\sqrt{u^2+a}} = \ln |u + \sqrt{u^2+a}| + C.$$

Sveska je skinuta sa stranice pf.unze.ba/nabokov

U svesci je moguća pojava grešaka - za uočene greške pisati na infoarrt@gmail.com